On dynamical r-matrices obtained from Dirac reduction and their generalizations to affine Lie algebras

2001 J. Phys. A: Math. Gen. 34 7235

(http://iopscience.iop.org/0305-4470/34/36/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7235–7248 PII: S0305-4470(01)25946-0

On dynamical r-matrices obtained from Dirac reduction and their generalizations to affine Lie algebras

L Fehér[1], **A Gábor and B G Pusztai**

Department of Theoretical Physics, University of Szeged, Tisza Lajos krt 84-86, H-6720 Szeged, Hungary

E-mail: lfeher@sol.cc.u-szeged.hu

Received 6 June 2001, in final form 8 August 2001
Published 31 August 2001
Online at stacks.iop.org/JPhysA/34/7235

Abstract

According to Etingof and Varchenko, the classical dynamical Yang–Baxter equation is a guarantee for the consistency of the Poisson bracket on certain Poisson–Lie groupoids. Here it is noticed that Dirac reductions of these Poisson manifolds give rise to a mapping from dynamical r-matrices on a pair $\mathcal{L} \subset \mathcal{A}$ to those on another pair $\mathcal{K} \subset \mathcal{A}$, where $\mathcal{K} \subset \mathcal{L} \subset \mathcal{A}$ is a chain of Lie algebras for which $\mathcal{L}$ admits a reductive decomposition as $\mathcal{L} = \mathcal{K}+\mathcal{M}$. Several known dynamical r-matrices appear naturally in this setting, and its application provides new r-matrices, too. In particular, we exhibit a family of r-matrices for which the dynamical variable lies in the grade zero subalgebra of an extended affine Lie algebra obtained from a twisted loop algebra based on an arbitrary finite-dimensional self-dual Lie algebra.

PACS numbers: 02.20.-a, 02.10.Yn

1. Introduction

The Yang–Baxter equation and the associated algebraic structures play a central role in the theory of integrable systems. Recently there has been growing interest in dynamical generalizations of these objects (for a review, see [1]). Our concern in this paper is the classical dynamical Yang–Baxter equation (CDYBE) that originally appeared in studies of the Liouville–Toda and the WZNW conformal field theories [2–4]. In its general form the CDYBE is defined [5] as follows. Let $\mathcal{A}$ be a Lie algebra and $\mathcal{L} \subset \mathcal{A}$ a Lie subalgebra with dual space

[1] Corresponding author.

$\mathcal{L}^*$. A dynamical r-matrix with respect to the pair $\mathcal{L} \subset \mathcal{A}$ is a map r from an open domain $\check{\mathcal{L}}^* \subset \mathcal{L}^*$ to $\mathcal{A} \otimes \mathcal{A}$ subject to

$$[r_{12}, r_{13}] + L_a^1 \frac{\partial r_{23}}{\partial \lambda_a} + \text{cycl. perm.} = 0. \tag{1.1}$$

Here the λ_a are coordinates on $\mathcal{L}^*$ with respect to a basis $\{L_a\}$ of $\mathcal{L}$; the usual tensorial notation and the summation convention are used throughout the paper. The cyclic permutations act on the three tensorial factors; for any $r = X^i \otimes Y_i \in \mathcal{A} \otimes \mathcal{A}$ one defines $r_{12} = X^i \otimes Y_i \otimes 1$, $r_{31} = Y_i \otimes 1 \otimes X^i$ and so on. It is further required that the symmetric part of r is an $\mathcal{A}$-invariant constant element of $\mathcal{A} \otimes \mathcal{A}$ and the function $r : \check{\mathcal{L}}^* \to \mathcal{A} \otimes \mathcal{A}$ is equivariant with respect to the natural infinitesimal actions of $\mathcal{L}$ on the respective spaces. Etingof and Varchenko [5] found an interesting geometric interpretation of the CDYBE that generalizes Drinfeld's interpretation of the CYBE in terms of Poisson–Lie groups [6], namely, they constructed a so-called dynamical Poisson–Lie groupoid structure on the direct product manifold

$$\check{\mathcal{L}}^* \times A \times \check{\mathcal{L}}^* \tag{1.2}$$

where A is a connected Lie group with Lie algebra $\mathcal{A}$. The Poisson structure on (1.2) is encoded by r in such a way that the antisymmetry and the Jacobi identity enforce the above-mentioned invariance and equivariance properties of r together with the condition that the function

$$\text{CDYB}(r) := [r_{12}, r_{13}] + L_a^1 \frac{\partial r_{23}}{\partial \lambda_a} + \text{cycl. perm.} \tag{1.3}$$

must yield an A-invariant constant element of $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}$. Many examples and a classification of the meromorphic classical dynamical r-matrices for certain choices of the pair $\mathcal{L} \subset \mathcal{A}$ are now available [1, 5].

The purpose of this paper is to point out a simple mechanism whereby some known and some new solutions of the CDYBE can be viewed from a unified perspective. Our basic idea is that the imposition of suitable constraints on the dynamical Poisson–Lie groupoid (1.2) will result in a reduced Poisson–Lie groupoid of the form

$$\check{\mathcal{K}}^* \times A \times \check{\mathcal{K}}^* \tag{1.4}$$

for some subalgebras $\mathcal{K} \subset \mathcal{L}$. The Dirac bracket defined by the reduction will be encoded by an r-matrix $r^* : \check{\mathcal{K}}^* \to \mathcal{A} \otimes \mathcal{A}$ that solves the CDYBE for the pair $\mathcal{K} \subset \mathcal{A}$ whenever the original r-matrix solves it for the pair $\mathcal{L} \subset \mathcal{A}$. It will be shown that the reduction works in this manner if $\mathcal{K} \subset \mathcal{L}$ admits a $\mathcal{K}$ invariant complementary linear space and the constraints of the reduction are second class. Under these conditions, we obtain a simple formula for r^* by applying the standard formula to determine the Dirac bracket on $\check{\mathcal{K}}^* \times A \times \check{\mathcal{K}}^*$. This formula implies that $\text{CDYB}(r) = \text{CDYB}(r^*)$, and therefore the reduction closes on classical dynamical r-matrices.

Our remark on the Dirac reduction of dynamical r-matrices complements the known constructions of solutions of the CDYBE and sheds a new light on the origin of some solutions. For instance, if the pair $\mathcal{L} \subset \mathcal{A}$ is given by the Cartan subalgebra of a simple Lie algebra, which is a case of principal interest, then the corresponding basic rational and trigonometric solutions can be viewed as Dirac reductions of respectively the zero and the so-called 'canonical' (or Alekseev–Meinrenken) r-matrices [5, 7, 8] for which $\mathcal{L} = \mathcal{A}$. We note that an equivalent result can be extracted from [5] as well (see theorem 3.14 in [5]). However, Dirac reduction is not mentioned in [5], and it works in circumstances more general than those considered in this reference. In particular, in equation (4.7) a class of r-matrices is displayed which is applicable to arbitrary (not necessarily simple or reductive) finite-dimensional self-dual Lie algebras [9]. To illustrate that formula (4.7) contains new dynamical r-matrices, too, we shall apply it to the self-dual extension [10] of the Euclidean Lie algebra $e(d)$ for even d. Moreover, we shall show that this formula remains well defined in certain infinite-dimensional situations as well. In fact, several new r-matrices will be obtained by applying (4.7) in the cases for which the dynamical variable lies in the grade zero subalgebra of an extended affine Lie algebra associated with a twisted loop algebra based on an arbitrary finite-dimensional self-dual Lie algebra. These yield generalizations of Felder's spectral-parameter-dependent dynamical r-matrices [4] upon applying evaluation homomorphisms to the twisted loop algebras.

The organization of the paper is the following. A short recall of the geometric interpretation of the CDYBE from [5] is presented in section 2. Section 3 is devoted to the Dirac reduction of dynamical r-matrices. In section 4 examples are given on arbitrary finite-dimensional self-dual Lie algebras, and some of these r-matrices are generalized to affine Lie algebras in section 5. The final section contains a discussion of the results, open questions and comments on the literature.

The main results are given by proposition 1 in section 3, equation (4.7) in section 4 and proposition 2 in section 5. We consider the development of the Dirac reduction viewpoint to be our most important result, since it may lead to further results in the future. For example, it should be possible to apply Hamiltonian reduction *after* quantizing the Poisson–Lie groupoids that underlie the dynamical r-matrices, since the second-class constraints that appear in the examples usually admit a natural separation into first-class constraints and gauge fixing conditions.

2. Geometric interpretation of the CDYBE

We wish to apply Dirac reduction to the dynamical Poisson–Lie groupoids that encode the dynamical r-matrices. As a preparation, we here recall from [5] the definition of these Poisson manifolds in a form convenient for our purpose.

Let us denote the elements of the space in (1.2) as

$$\check{\mathcal{L}}^* \times A \times \check{\mathcal{L}}^* = \{(\lambda^F, g, \lambda^I)\} \tag{2.1}$$

and let $\lambda_a := \lambda(L_a)$ be the components of $\lambda \in \check{\mathcal{L}}^*$ with respect to a basis L_a of $\mathcal{L}$ for which

$$[L_a, L_b] = f_{ab}{}^c L_c. \tag{2.2}$$

Consider a function $r : \check{\mathcal{L}}^* \to \mathcal{A} \otimes \mathcal{A}$, and equip $\check{\mathcal{L}}^* \times A \times \check{\mathcal{L}}^*$ with a Poisson bracket $\{\ ,\ \}$ of the following form:

$$\begin{aligned}
&\{g_1, g_2\} = g_1 g_2 r(\lambda^I) - r(\lambda^F) g_1 g_2 \\
&\{g, \lambda_a^I\} = g L_a \\
&\{g, \lambda_a^F\} = L_a g \\
&\{\lambda_a^I, \lambda_b^I\} = -f_{ab}{}^c \lambda_c^I \\
&\{\lambda_a^F, \lambda_b^F\} = f_{ab}{}^c \lambda_c^F \\
&\{\lambda_a^I, \lambda_b^F\} = 0.
\end{aligned} \tag{2.3}$$

In this formula $g_1 := g \otimes 1$ and $g_2 := 1 \otimes g$ are really defined in terms of matrix representations of the group A. If one fixes a representation, then the first line of (2.3) serves to define the value of $\{g_1, g_2\}_{ij,kl} = \{g_{ij}, g_{kl}\}$, while the second line means that $\{g_{ij}, \lambda_a^I\} = (g L_a)_{ij}$. The antisymmetry and the Jacobi identity of the Poisson bracket lead to the requirements on r mentioned in the introduction as follows [5]. First, the antisymmetry $\{g_1, g_2\} = -\{g_2, g_1\}$ requires

$$r^s := \tfrac{1}{2}(r + r_{21}) \tag{2.4}$$

to be an A-invariant constant element of $\mathcal{A} \otimes \mathcal{A}$. Second, the Jacobi identities

$$\{\{g_1, g_2\}, \lambda_a^I\} + \text{cycl. perm.} = 0 = \{\{g_1, g_2\}, \lambda_a^F\} + \text{cycl. perm.} \tag{2.5}$$

are equivalent to the condition

$$[L_a^1 + L_a^2, r(\lambda)] = f_{ba}{}^c \lambda_c \frac{\partial r(\lambda)}{\partial \lambda_b} \tag{2.6}$$

which is the coordinatewise description of the $\mathcal{L}$-equivariance of the map r. This equation further restricts only the antisymmetric part r^a of $r = (r^s + r^a)$. Third, an easy calculation gives

$$\{\{g_1, g_2\}, g_3\} + \text{cycl. perm.} = \big(\text{CDYB}(r)(\lambda^F)\big)G - G\big(\text{CDYB}(r)(\lambda^I)\big) \tag{2.7}$$

with $G := g \otimes g \otimes g$. This means that $\text{CDYB}(r)$ must be an A-invariant constant element of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

We saw that (2.3) is indeed a Poisson bracket if and only if r^s and $\text{CDYB}(r)$ are A-invariant constants and (2.6) holds. If r^s is an A-invariant constant, then

$$\text{CDYB}(r^a + r^s) = \text{CDYB}(r^a) + \text{CDYB}(r^s) = \text{CDYB}(r^a) + [r_{12}^s, r_{13}^s]. \tag{2.8}$$

One sees from this that $\text{CDYB}(r)$ belongs to $\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \subset \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. Clearly, r^s drops out from the Poisson bracket (2.3). Its sole role is that in many cases one can achieve $\text{CDYB}(r) = 0$ by adding a suitable r^s to an r^a for which $\text{CDYB}(r^a)$ is a nonzero constant.

Below we use only the above-mentioned features of the Poisson manifold (2.1). The form of the Poisson bracket (2.3) guarantees that (2.1) is a Poisson–Lie groupoid in the sense of Weinstein [11]. This is readily verified from the definitions, but is not directly relevant for the purposes of this paper (see [5]). Note that the Poisson bracket (2.3) is also valid in the trivial case for which $r = 0$, and we shall see that the Dirac reduction of this case leads to dynamical r-matrices for which $\text{CDYB}(r^a) = 0$.

3. Dirac reduction acting on dynamical r-matrices

We wish to reduce the phase space (2.1), (2.3) to an object of a similar kind (1.4) with respect to a subalgebra $\mathcal{K} \subset \mathcal{L}$. For the reduction to work, we need two assumptions. The first assumption is that $\mathcal{K}$ admits an invariant complementary linear space $\mathcal{M}$ in $\mathcal{L}$, that is we have

$$\mathcal{L} = \mathcal{K} + \mathcal{M} \qquad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}. \tag{3.1}$$

In this case we can choose an adapted basis of $\mathcal{L}$ as

$$\{L_a\} = \{K_i\} \cup \{M_\alpha\} \qquad K_i \in \mathcal{K} \quad M_\alpha \in \mathcal{M}. \tag{3.2}$$

Correspondingly, the structure constants of $\mathcal{L}$ become

$$[K_i, K_j] = f_{ij}{}^k K_k \qquad [K_i, M_\alpha] = f_{i\alpha}{}^\beta M_\beta \qquad [M_\alpha, M_\beta] = f_{\alpha\beta}{}^\gamma M_\gamma + f_{\alpha\beta}{}^i K_i. \tag{3.3}$$

We also have the induced decomposition

$$\mathcal{L}^* = \mathcal{K}^* + \mathcal{M}^* \qquad \mathcal{K}^* := \mathcal{M}^\perp \quad \mathcal{M}^* := \mathcal{K}^\perp. \tag{3.4}$$

Accordingly, we decompose any $\lambda \in \mathcal{L}^*$ as

$$\lambda = \kappa + \mu \qquad \text{with} \quad \kappa \in \mathcal{K}^* \quad \mu \in \mathcal{M}^* \tag{3.5}$$

and these constituents have the components

$$\kappa_i = \kappa(K_i) = \lambda(K_i) = \lambda_i \qquad \text{and} \qquad \mu_\alpha = \mu(M_\alpha) = \lambda(M_\alpha) = \lambda_\alpha. \tag{3.6}$$

We define the reduction by putting the $\mathcal{M}^*$-components of λ^I and λ^F to zero; i.e., we impose the constraints

$$\lambda^I_\alpha = 0 \qquad \text{and} \qquad \lambda^F_\alpha = 0. \tag{3.7}$$

We want these constraints to be second class in the Dirac sense [12]. Clearly, this means that the function

$$\mathcal{C}_{\alpha\beta}(\kappa) := -f_{\alpha\beta}{}^i \kappa_i \tag{3.8}$$

must define an invertible matrix. Our second assumption is that this condition holds after a possible restriction of the domain $\check{\mathcal{L}}^*$. More precisely, we assume that

$$\check{\mathcal{K}}^* := \{\kappa \in \check{\mathcal{L}}^* \cap \mathcal{K}^* \mid \mathcal{C}(\kappa) : \text{invertible}\} \neq \emptyset \tag{3.9}$$

i.e. that $\check{\mathcal{K}}^*$ is a *nonempty* open submanifold of $\mathcal{K}^*$. The inverse of the matrix $\mathcal{C}_{\alpha\beta}(\kappa)$ will be denoted by $\mathcal{D}^{\alpha\beta}(\kappa)$,

$$\mathcal{C}_{\alpha\beta}(\kappa)\mathcal{D}^{\beta\gamma}(\kappa) = \delta^\gamma_\alpha. \tag{3.10}$$

Under these assumptions, the constrained manifold

$$\check{\mathcal{K}}^* \times A \times \check{\mathcal{K}}^* := \{(\kappa^F, g, \kappa^I)\} \tag{3.11}$$

is equipped with an induced Poisson bracket $\{,\}^*$ given by the application of Dirac's well known formula. For functions F_1 and F_2 on $\check{\mathcal{K}}^* \times A \times \check{\mathcal{K}}^*$, we have

$$\{F_1, F_2\}^* = \{\tilde{F}_1, \tilde{F}_2\} - \{\tilde{F}_1, \lambda^I_\alpha\}\mathcal{D}^{\alpha\beta}(\kappa^I)\{\lambda^I_\beta, \tilde{F}_2\} + \{\tilde{F}_1, \lambda^F_\alpha\}\mathcal{D}^{\alpha\beta}(\kappa^F)\{\lambda^F_\beta, \tilde{F}_2\} \tag{3.12}$$

where the $\tilde{F}_i$ are arbitrary extensions of the F_i to a neighbourhood of the constrained manifold in $\check{\mathcal{L}}^* \times A \times \check{\mathcal{L}}^*$ and the function on the right-hand side is restricted to $\check{\mathcal{K}}^* \times A \times \check{\mathcal{K}}^*$ after the evaluation of the Poisson brackets. Convenient extensions are provided by requiring the $\tilde{F}_i$ to be independent of μ^I and μ^F defined by (3.5). Proceeding in this manner, we easily find the following Dirac brackets:

$$\begin{aligned}
&\{g_1, g_2\}^* = g_1 g_2 r^*(\kappa^I) - r^*(\kappa^F) g_1 g_2\\
&\{g, \kappa^I_i\}^* = g K_i\\
&\{g, \kappa^F_i\}^* = K_i g\\
&\{\kappa^I_i, \kappa^I_j\}^* = -f_{ij}{}^k \kappa^I_k\\
&\{\kappa^F_i, \kappa^F_j\}^* = f_{ij}{}^k \kappa^F_k\\
&\{\kappa^I_i, \kappa^F_j\}^* = 0
\end{aligned} \tag{3.13}$$

where

$$r^*(\kappa) := r(\kappa) + \mathcal{D}^{\alpha\beta}(\kappa) M_\alpha \otimes M_\beta \qquad \forall \kappa \in \check{\mathcal{K}}^*. \tag{3.14}$$

Notice that the Dirac brackets that involve the components of κ^I or κ^F are 'the same' as the corresponding original Poisson brackets. This is guaranteed by (3.12) upon using (3.1), which explains why this assumption was made. For later reference, denote the restriction of $r : \check{\mathcal{L}}^* \to \mathcal{A} \otimes \mathcal{A}$ to $\check{\mathcal{K}}^*$ by $\tilde{r}$ and introduce the map $\mathcal{D} : \check{\mathcal{K}}^* \to \mathcal{A} \otimes \mathcal{A}$ in correspondence with the second term in (3.14). In this notation,

$$r^* = \tilde{r} + \mathcal{D}. \tag{3.15}$$

It is obvious that the symmetric part of r^* is equal to the symmetric part of r, which is an A-invariant constant. The $\mathcal{K}$-equivariance of the map $r^* : \check{\mathcal{K}}^* \to \mathcal{A} \otimes \mathcal{A}$ is guaranteed since the Dirac bracket satisfies the Jacobi identity, and one can also directly check this equivariance property:

$$[K^1_i + K^2_i, r^*(\kappa)] = f_{ji}{}^k \kappa_k \frac{\partial r^*(\kappa)}{\partial \kappa_j}. \tag{3.16}$$

For the same reason, it follows that

$$\mathrm{CDYB}(r^*) := [r^*_{12}, r^*_{13}] + K^1_i \frac{\partial r^*_{23}}{\partial \kappa_i} + \text{cycl. perm.} \tag{3.17}$$

defines an A-invariant constant element of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$. One may expect this constant to be the same as the constant given by $\mathrm{CDYB}(r)$, which is determined by the formula (1.3). Indeed, we can verify the following statement.

Proposition 1. *Consider an $\mathcal{L}$-equivariant map $r : \check{\mathcal{L}}^* \to \mathcal{A}\otimes\mathcal{A}$ and suppose that equations (3.1) and (3.9) hold. Then one has the equalities*

$$\mathrm{CDYB}(\mathcal{D}) = 0 \qquad \mathrm{CDYB}(r^*) = \mathrm{CDYB}(r) \tag{3.18}$$

where $r^* : \check{\mathcal{K}}^* \to \mathcal{A}\otimes\mathcal{A}$ and $\mathcal{D} : \check{\mathcal{K}}^* \to \mathcal{M}\otimes\mathcal{M}$ are given by (3.14) with (3.10).

This statement and its interpretation in terms of Dirac reduction represent the first main result of the present paper. The first equality means that under the assumptions in (3.1) and (3.9) the map $\mathcal{D} : \check{\mathcal{K}}^* \to \mathcal{A}\otimes\mathcal{A}$ is an antisymmetric solution of the CDYBE for the pair $\mathcal{K}\subset\mathcal{A}$. More precisely, since $\mathcal{D}(\check{\mathcal{K}}^*) \subset \mathcal{L}\otimes\mathcal{L}$, this is a solution of the CDYBE for the pair $\mathcal{K}\subset\mathcal{L}$. The second equality implies that if r is a classical dynamical r-matrix for the pair $\mathcal{L}\subset\mathcal{A}$ then so is r^* for the pair $\mathcal{K}\subset\mathcal{A}$. In the special case for which $\mathcal{K}$ is a Cartan subalgebra and $\mathcal{L}$ is a reductive subalgebra of a simple Lie algebra, the statement of the proposition has been proved in [5]. In fact, the proof that we present is analogous to the proof of theorem 3.14 in [5], but we use only the assumptions in (3.1) and (3.9) without any other special features of the Lie algebras $\mathcal{K}\subset\mathcal{L}\subset\mathcal{A}$.

In order to verify that $\mathrm{CDYB}(\mathcal{D}) = 0$, first note that

$$K^1_i \frac{\partial \mathcal{D}_{23}(\kappa)}{\partial \kappa_i} = -f_{\gamma\theta}{}^{i}\mathcal{D}^{\gamma\alpha}(\kappa)\mathcal{D}^{\theta\beta}(\kappa) K_i\otimes M_\alpha\otimes M_\beta \tag{3.19}$$

which follows by computing the derivatives on account of (3.10) and (3.8). By using this, we find that

$$\mathrm{CDYB}(\mathcal{D}) = Q^{\nu\alpha\beta} M_\nu\otimes M_\alpha\otimes M_\beta \tag{3.20}$$

with

$$Q^{\nu\alpha\beta} = f_{\gamma\theta}{}^{\nu}\mathcal{D}^{\gamma\alpha}\mathcal{D}^{\theta\beta} + f_{\gamma\theta}{}^{\beta}\mathcal{D}^{\gamma\nu}\mathcal{D}^{\theta\alpha} + f_{\gamma\theta}{}^{\alpha}\mathcal{D}^{\gamma\beta}\mathcal{D}^{\theta\nu}. \tag{3.21}$$

Multiplying by invertible matrices, we then obtain

$$Q^{\nu\alpha\beta}\mathcal{C}_{\alpha\xi}\mathcal{C}_{\beta\eta} = f_{\xi\eta}{}^{\nu} - \mathcal{D}^{\gamma\nu}(f_{\eta\gamma}{}^{a}f_{a\xi}{}^{i} + f_{\gamma\xi}{}^{a}f_{a\eta}{}^{i})\kappa_i = f_{\xi\eta}{}^{\nu} + \mathcal{D}^{\gamma\nu}f_{\xi\eta}{}^{a}f_{a\gamma}{}^{i}\kappa_i = 0. \tag{3.22}$$

For the first equality, we used that $f_{\eta\gamma}{}^{a}f_{a\xi}{}^{i} = f_{\eta\gamma}{}^{\alpha}f_{\alpha\xi}{}^{i}$, where the indices a and α run over the bases of $\mathcal{L}$ and $\mathcal{M}$, respectively, and the equality holds because of (3.1). The second equality is valid on account of the Jacobi identity for $\mathcal{L}$, while the third equality is implied by the definitions of $\mathcal{C}$ and $\mathcal{D}$. Since we have shown that $Q^{\nu\alpha\beta} = 0$, $\mathrm{CDYB}(\mathcal{D}) = 0$ follows by (3.20).

We start the proof of the second equality in (3.18) by remarking that

$$[M^2_\alpha + M^3_\alpha, r_{23}(\kappa)] = \mathcal{C}_{\alpha\beta}(\kappa)\frac{\partial r_{23}}{\partial \lambda_\beta}(\kappa). \tag{3.23}$$

This follows from (2.6) upon imposing the constraint $\kappa = \mu \in \check{\mathcal{K}}^*$. This equality then implies that

$$M^1_\alpha \frac{\partial r_{23}}{\partial \lambda_\alpha}(\kappa) = [\mathcal{D}_{12}(\kappa) + \mathcal{D}_{23}(\kappa), r_{23}(\kappa)]. \tag{3.24}$$

By using this and $\mathrm{CDYB}(\mathcal{D}) = 0$, it is easy to obtain from (3.15) that

$$[r_{12}, r_{13}](\kappa) + L^1_a \frac{\partial r_{23}}{\partial \lambda_a}(\kappa) + \text{cycl. perm.} = [r^*_{12}(\kappa), r^*_{23}(\kappa)] + K^1_i \frac{\partial r^*_{23}}{\partial \kappa_i}(\kappa) + \text{cycl. perm.} \tag{3.25}$$

whereby the proof is complete.

For any constant, nonzero, A-invariant element $\varphi \in \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}$, a modified version of the CDYBE may be defined by replacing the zero on the right-hand side of (1.1) with φ. It is clear from proposition 1 that the Dirac reduction maps not only the solutions of the CDYBE but also the solutions of this modified CDYBE with respect to $\mathcal{L} \subset \mathcal{A}$ to those with respect to $\mathcal{K} \subset \mathcal{A}$, for any fixed invariant φ.

4. Examples on self-dual Lie algebras

The examples that we describe next are obtained in the situation for which the Lie algebra $\mathcal{A}$ admits a nondegenerate, invariant symmetric bilinear form[2] $\langle\ ,\ \rangle$ that remains nondegenerate upon restriction to the subalgebras $\mathcal{K}$ and $\mathcal{L}$ in the chain $\mathcal{K} \subset \mathcal{L} \subset \mathcal{A}$. The bilinear form induces the identifications $\mathcal{A}^* = \mathcal{A}$, $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{K}^* = \mathcal{K}$ and allows one to associate with any element of $\mathcal{A} \otimes \mathcal{A}$ a linear operator on $\mathcal{A}$; the operator associated with $X \otimes Y$ sends Z to $\langle Y, Z\rangle X$ for any $X, Y, Z \in \mathcal{A}$. The assumption in (3.1) is now guaranteed if we let

$$\mathcal{M} := \mathcal{K}^\perp_{\mathcal{L}} := \{\lambda \in \mathcal{L} \mid \langle \lambda, \kappa\rangle = 0\ \forall \kappa \in \mathcal{K}\}. \tag{4.1}$$

Let us now suppose that the invertibility assumption (3.9) holds and denote the $\mathrm{End}(\mathcal{A})$-valued functions associated with r and r^* by ρ and ρ^*, respectively. Then formula (3.15) can be rewritten in the form

$$\rho^*(\kappa)(X) = \begin{cases} \rho(\kappa)(X) & \text{if} \quad X \in (\mathcal{K} + \mathcal{L}^\perp) \\ \rho(\kappa)(X) + \left((\mathrm{ad}\,\kappa)|_{\mathcal{K}^\perp_{\mathcal{L}}}\right)^{-1}(X) & \text{if} \quad X \in \mathcal{K}^\perp_{\mathcal{L}}. \end{cases} \tag{4.2}$$

The domain $\check{\mathcal{K}}$ consists of those elements $\kappa \in \check{\mathcal{L}} \cap \mathcal{K}$ for which the restriction of the operator $\mathrm{ad}\,\kappa$ to $\mathcal{K}^\perp_{\mathcal{L}}$ is invertible. To obtain concrete examples, we have to start with a dynamical r-matrix $\rho : \check{\mathcal{L}} \to \mathrm{End}(\mathcal{A})$ and have to ensure that $\check{\mathcal{K}}$ is nonempty.

If we start with the trivial (zero) r-matrix, then *(4.2) with $\rho = 0$ provides an antisymmetric solution of the CDYBE whenever $\check{\mathcal{K}} \subset \mathcal{K}$ is nonempty.* Although this remark appears quite trivial, many antisymmetric solutions of the CDYBE can be understood as its special cases. For example, theorem 3.2 of [5] implies that if one takes $\mathcal{K}$ to be a Cartan subalgebra of a simple Lie algebra $\mathcal{A}$ and lets $\mathcal{L}$ vary, then one can recover from (4.2) essentially (i.e. up to some obvious gauge transformations) all antisymmetric solutions of the CDYBE for the pair $\mathcal{K} \subset \mathcal{A}$.

Somewhat more interestingly, we may also take as our starting point a 'canonical' dynamical r-matrix that is available in the case $\mathcal{L} = \mathcal{A}$ for *any* self-dual Lie algebra $\mathcal{A}$. This r-matrix is defined by using the holomorphic complex function

$$f(z) := \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z}. \tag{4.3}$$

It was found in [5, 7, 8] (see also [13, 14]) that the r-matrix associated with the linear operator

$$\rho_\pm(\lambda) = f(\mathrm{ad}\,\lambda) \pm \tfrac{1}{2} I \qquad \lambda \in \check{\mathcal{A}} \tag{4.4}$$

[2] Such Lie algebras, which include, for example, the reductive Lie algebras and the Drinfeld doubles of the Lie bialgebras, are called self-dual in this paper. For their structure, one may consult [9].

solves the CDYBE (1.1) for $\mathcal{L} = \mathcal{A}$. In the standard manner (see e.g. [15], chapter 7), the holomorphic function f can be applied to the operator ad λ with the aid of the formula

$$f(\mathrm{ad}\,\lambda) := \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}z\, f(z)(zI - \mathrm{ad}\,\lambda)^{-1} \tag{4.5}$$

where Γ is a contour that encircles each eigenvalue of ad λ. This expression is well defined and is independent of the contour Γ if f is holomorphic in a neighbourhood of the spectrum σ_λ of ad λ. Thus the domain $\check{\mathcal{A}}$ in this case is naturally specified as

$$\check{\mathcal{A}} = \{\lambda \in \mathcal{A} \,|\, 2\pi\mathrm{i}n \notin \sigma_\lambda\ \forall n \in \mathbb{Z}^*\} \qquad (\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}). \tag{4.6}$$

The definition of $f(\mathrm{ad}\,\lambda)$ in (4.5) is equivalent to the alternative definition by means of the Taylor series expansion of $f(z)$ around $z = 0$, which is applicable if σ_λ lies inside the disc on which that series converges [15].

In terms of the corresponding linear operators $\rho^*_\pm$, the reduction of the r-matrix (4.4) to a self-dual subalgebra $\mathcal{K} \subset \mathcal{A}$ (on which $\langle\ ,\ \rangle$ remains nondegenerate) can now be written as follows:

$$\rho^*_\pm(\kappa) = \begin{cases} f(\mathrm{ad}\,\kappa) \pm \frac{1}{2}I & \text{on} \quad \mathcal{K} \\ \frac{1}{2}\coth\left(\frac{1}{2}\mathrm{ad}\,\kappa\right) \pm \frac{1}{2}I & \text{on} \quad \mathcal{K}^\perp_{\mathcal{A}}. \end{cases} \tag{4.7}$$

In this equation $\frac{1}{2}\coth(\frac{1}{2}\mathrm{ad}\,\kappa) = f(\mathrm{ad}\,\kappa) + (\mathrm{ad}\,\kappa)^{-1}$ on $\mathcal{K}^\perp_{\mathcal{A}}$ with f in (4.3). Hence *the operators* $\rho^*_\pm(\kappa)$ *are well defined on* $\mathcal{A}$ *if and only if the spectrum of* ad κ*, acting on* $\mathcal{A}$*, does not intersect* $2\pi\mathrm{i}\mathbb{Z}^*$*, and* $(\mathrm{ad}\,\kappa)|_{\mathcal{K}^\perp_{\mathcal{A}}}$ *is invertible. These are the conditions that the domain* $\check{\mathcal{K}} \subset \mathcal{K}$ *must satisfy.* The invertibility requirement on $(\mathrm{ad}\,\kappa)|_{\mathcal{K}^\perp_{\mathcal{A}}}$ means that one must restrict the original domain $\check{\mathcal{A}}$ for the reduction to be performable. Since $\check{\mathcal{K}}$ does not contain the zero element, its nonemptyness is a nontrivial condition on the subalgebra $\mathcal{K} \subset \mathcal{A}$. To state the result in the alternative tensorial terms, if the above conditions are satisfied, then solutions of the CDYBE for $\mathcal{K} \subset \mathcal{A}$ are provided by the functions $r^*_\pm : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ given by

$$r^*_\pm(\kappa) = \pm\frac{1}{2}\hat{I} + \langle K^i, f(\mathrm{ad}\,\kappa)K^j\rangle K_i \otimes K_j + \left\langle M^\alpha, \frac{1}{2}\coth\left(\frac{\mathrm{ad}\,\kappa}{2}\right) M^\beta\right\rangle M_\alpha \otimes M_\beta. \tag{4.8}$$

Here K_i, K^j and M_α, M^β denote dual bases of $\mathcal{K}$ and $\mathcal{K}^\perp_{\mathcal{A}}$, respectively, $\langle K_i, K^j\rangle = \delta_i^j$ and $\langle M_\alpha, M^\beta\rangle = \delta_\alpha^\beta$ and $\hat{I} := K_i \otimes K^i + M_\alpha \otimes M^\alpha$.

As we shall see below, many known solutions of the CDYBE can be recovered as special cases of (4.7), (4.8). The class of r-matrices given by these equations apparently has not been displayed before in this general form; its full set of special cases is still to be uncovered. It can be checked independently of the Dirac reduction argument, too, that (4.7) provides a solution of the CDYBE whenever one has a decomposition $\mathcal{A} = \mathcal{K} + \mathcal{K}^\perp_{\mathcal{A}}$ such that the above formula yields a well defined linear operator on $\mathcal{A}$. This is important in view of interesting new examples for which $\mathcal{A}$ is infinite dimensional with a finite-dimensional $\mathcal{K}$ (see section 5).

Several solutions of the CDYBE can be obtained from (4.8) by taking $\mathcal{K} := \mathcal{A}_0$ with respect to an integral gradation $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ of $\mathcal{A}$ for which $\mathcal{K}^\perp_{\mathcal{A}} = \oplus_{n\in\mathbb{Z}^*}\mathcal{A}_n$. One must choose $\mathcal{A}$ and the gradation in such a way that a nonempty domain $\check{\mathcal{K}} \subset \mathcal{K}$ exists on which the components of $r^*_\pm$ are smooth functions. Notice that this is automatically guaranteed if $\mathcal{A}$ is a complex simple Lie algebra. Indeed, in this case the regular semisimple elements form a dense open submanifold in $\mathcal{A}_0$ for any integral gradation, and thus ad κ is invertible on $\mathcal{K}^\perp_{\mathcal{A}} = \oplus_{n\in\mathbb{Z}^*}\mathcal{A}_n$ if κ belongs to a small ball in $\mathcal{K}$ around such a regular element. An alternative description of precisely these examples is provided by theorem 3.14 in [5]. We note in passing that in this case the second-class constraints of the Dirac reduction can be naturally separated into first-class constraints and gauge fixing conditions, simply by decomposing $\mathcal{K}^\perp_{\mathcal{A}}$ into positively and negatively graded subspaces.

To recover the basic trigonometric dynamical r-matrix from the above-mentioned examples, let us now take $\mathcal{A}$ to be a finite-dimensional complex simple Lie algebra equipped with the principal gradation and identify $\mathcal{K}$ with the Cartan subalgebra given by the grade zero elements. Then the M_α in (4.8) can be taken to be the root vectors associated with the set of roots Φ with respect to $\mathcal{K} \subset \mathcal{A}$, and (4.8) yields

$$r^*_\pm(\kappa) = \pm\frac{1}{2}\hat{I} + \sum_{\alpha\in\Phi} \frac{|\alpha|^2}{4} \coth\left(\frac{\alpha(\kappa)}{2}\right) M_\alpha \otimes M_{-\alpha}. \tag{4.9}$$

We here used that $[\kappa, M_\alpha] = \alpha(\kappa)M_\alpha$, $\langle M_\alpha, M_{-\alpha}\rangle = \frac{2}{|\alpha|^2}$ and that $f(\mathrm{ad}\,\kappa)K^j$ in (4.8) now vanishes since $\mathcal{K}$ is Abelian. This solution of the CDYBE (1.1) first appeared in studies of the WZNW and conformal Toda field theories [3]. It has been proved in [5] that if the dynamical variable belongs to a Cartan subalgebra of a simple Lie algebra then all solutions of (1.1) can be obtained from (4.9) by shifts of the argument κ by a constant and simple limiting procedures. Note in passing that the analogous reduction with $r = 0$ as starting point leads to the rational r-matrix [5] $r^*_0(\kappa) = \sum_{\alpha\in\Phi} \frac{|\alpha|^2}{2\alpha(\kappa)} M_\alpha \otimes M_{-\alpha}$.

It was found in [5] that the natural generalization of (4.9) defines a dynamical r-matrix also for an affine Kac–Moody Lie algebra $\mathcal{A}$. To obtain this generalization, one uses the principal gradation of $\mathcal{A}$ for which $\mathcal{K} := \mathcal{A}_0$ is the Cartan subalgebra, and correspondingly extends the summation in (4.9) over the roots of $\mathcal{A}$. Motivated by this result, in section 5 we display a large family of dynamical r-matrices on affine Lie algebras based on *arbitrary* finite-dimensional self-dual Lie algebras.

Before turning to infinite-dimensional Lie algebras, we wish to show that our general formula (4.7) contains new examples for finite-dimensional self-dual Lie algebras, too. To illustrate this, we now take $\mathcal{A}$ to be the well known [10] self-dual extension of the complex Euclidean Lie algebra $e(d)$. We denote the generators of $e(d)$ ($d \geqslant 2$) by P_i and J_{ij}, where $i, j = 1, \ldots, n$ and the relation $J_{ji} = -J_{ij}$ is understood. The J_{ij} span the orthogonal Lie algebra $o(d) \subset e(d)$, and we let T_{ij} ($T_{ji} = -T_{ij}$) denote the generators of the dual space of $o(d)$. By definition, $\mathcal{A} = \mathrm{span}\{P_i, J_{ij}, T_{ij}\}$ has the commutation relations

$$\begin{aligned}
&[J_{ij}, J_{kl}] = \delta_{jk}J_{il} + \delta_{il}J_{jk} - \delta_{jl}J_{ik} - \delta_{ik}J_{jl}\\
&[J_{ij}, T_{kl}] = \delta_{jk}T_{il} + \delta_{il}T_{jk} - \delta_{jl}T_{ik} - \delta_{ik}T_{jl}\\
&[T_{ij}, T_{kl}] = [T_{ij}, P_k] = 0\\
&[J_{ij}, P_k] = \delta_{jk}P_i - \delta_{ik}P_j\\
&[P_k, P_l] = T_{kl}.
\end{aligned} \tag{4.10}$$

There is a one-parameter family of invariant scalar products on $\mathcal{A}$, which in terms of our redundant set of generators is given by

$$\begin{aligned}
&\langle J_{ij}, J_{kl}\rangle = p(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})\\
&\langle J_{ij}, T_{kl}\rangle = \delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}\\
&\langle J_{ij}, P_k\rangle = \langle T_{ij}, P_k\rangle = 0\\
&\langle P_i, P_j\rangle = \delta_{ij}
\end{aligned} \tag{4.11}$$

where p is an arbitrary constant. It is clear that $\mathcal{K} := \mathrm{span}\{J_{ij}, T_{ij}\}$ is a self-dual subalgebra and $\mathcal{K}^\perp_\mathcal{A} = \mathrm{span}\{P_i\}$. By writing $\kappa \in \mathcal{K}$ as $\kappa = x^{ij}J_{ij} + y^{ij}T_{ij}$ and $P \in \mathcal{K}^\perp_\mathcal{A}$ as $P = z^iP_i$, where summation is understood and the components x^{ij}, y^{ij} are antisymmetric in ij, we see that $[\kappa, P] = 2\sum_{i,k} x^{ik}z^kP_i$. Since the determinant of an antisymmetric matrix of odd size is zero, $\mathrm{ad}\,\kappa$ is never invertible on $\mathcal{K}^\perp_\mathcal{A}$ if d is odd, and hence we do not obtain a nonempty domain $\check{\mathcal{K}}$ in this case. However, if d is even, then one may check that for $\kappa_0 := J_{12} + J_{34} + \cdots + J_{d-1,d}$ the operator $(\mathrm{ad}\,\kappa_0)|_{\mathcal{K}_{\mathcal{A}}^{\perp}}$ is invertible. It follows that for a small but nonzero constant q the element $q\kappa_0 \in \mathcal{K}$ satisfies the spectral conditions described below equation (4.7). This implies that $\check{\mathcal{K}} \subset \mathcal{K}$ is a *nonempty open domain for any even* d, and (4.7) provides us with new dynamical r-matrices in this case.

We note that for $d = 2$ (4.10) defines the central extension of $e(2)$ that has interesting physical applications. In this case $\mathcal{K}$ is a two-dimensional Abelian Lie algebra. Further examples for which $\mathcal{K}$ is two dimensional and Abelian can be obtained by taking $\mathcal{A}$ to be the oscillator Lie algebra generated by $a_i, a_i^\dagger$ $(i = 1, \ldots, n)$, the central element $\hat{c}$ and the number operator $\hat{N}$. With respect to the usual scalar product [16], $\mathcal{K} = \mathrm{span}\{\hat{N}, \hat{c}\}$ is a self-dual subalgebra and $\check{\mathcal{K}}$ is easily seen to be nonempty. In these cases, it should not be too difficult to quantize the above-constructed dynamical r-matrices.

5. Generalizations to affine Lie algebras

In this section we describe generalizations of the r-matrices that appear in (4.7) for situations in which the dynamical variable lies in a finite-dimensional subalgebra of an infinite-dimensional Lie algebra $\mathcal{A}$. In fact, we shall take $\mathcal{A}$ to be an 'affine Lie algebra' obtained by central extension and inclusion of the derivation from a twisted loop algebra built on a finite-dimensional self-dual Lie algebra $\mathcal{G}$, and let the dynamical variable lie in the grade zero part of $\mathcal{A}$.

We start with a preliminary remark that will be used below. Let $\mathcal{A}$ be a (possibly infinite-dimensional) self-dual Lie algebra with scalar product $\langle\ ,\ \rangle$. Consider a decomposition

$$\mathcal{A} = \mathcal{K} + \mathcal{K}^{\perp} \qquad \mathcal{K} \cap \mathcal{K}^{\perp} = \{0\} \tag{5.1}$$

where $\mathcal{K} \subset \mathcal{A}$ is a finite-dimensional self-dual Lie subalgebra. Let us now denote by $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ the operator-valued function corresponding to a function[3] $r : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$, where $\check{\mathcal{K}}$ is some open subset of $\mathcal{K}$. By assuming the existence of the directional derivative

$$(\nabla_T R)(\kappa) := \frac{\mathrm{d}}{\mathrm{d}t} R(\kappa + tT)|_{t=0} \qquad \forall T \in \mathcal{K} \quad \kappa \in \check{\mathcal{K}} \tag{5.2}$$

let us define

$$\langle X, (\nabla R)(\kappa) Y\rangle := \sum_i K^i \langle X, (\nabla_{K_i} R)(\kappa) Y\rangle \qquad \forall X, Y \in \mathcal{A} \tag{5.3}$$

where $\{K_i\}$ and $\{K^i\}$ are dual bases of $\mathcal{K}$, $\langle K_i, K^j\rangle = \delta_i^j$. Denote by $\hat{f} \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ the (antisymmetric) invariant element associated with the Lie bracket of $\mathcal{A}$, and $\hat{I} \in \mathcal{A} \otimes \mathcal{A}$ the (symmetric) invariant element associated with the unit operator on $\mathcal{A}$. (If T_α and T^α are dual bases of $\mathcal{A}$ and $[T_\alpha, T_\beta] = f_{\alpha\beta}^\gamma T_\gamma$, then $\hat{f} = f_{\alpha\beta}^\gamma T^\alpha \otimes T^\beta \otimes T_\gamma$ and $\hat{I} = T_\alpha \otimes T^\alpha$.) We have the following lemma.

Lemma. *Let us consider an antisymmetric r-matrix* $r : \check{\mathcal{K}} \to \mathcal{A} \wedge \mathcal{A}$ *and the associated operator* $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$. *Then the equation*

$$\mathrm{CDYB}(r) = -C^2 \hat{f} \tag{5.4}$$

where C is some complex constant, is equivalent to

$$\begin{aligned}&[RX, RY] - R([X, RY] + [RX, Y]) + \langle X, (\nabla R) Y\rangle + (\nabla_{Y_{\mathcal{K}}} R) X - (\nabla_{X_{\mathcal{K}}} R) Y \\ &\qquad = -C^2 [X, Y] \qquad \forall X, Y \in \mathcal{A}.\end{aligned} \tag{5.5}$$

[3] If $\mathcal{A}$ is infinite dimensional, then $\mathcal{A} \otimes \mathcal{A}$ denotes a certain completion of the algebraic tensor product, which is such that the corresponding linear operators are well defined on $\mathcal{A}$.

The statement of the lemma is straightforward to verify. Note that in (5.5) we use the decomposition $X = X_{\mathcal{K}} + X_{\mathcal{K}^\perp}$ with $X_{\mathcal{K}} \in \mathcal{K}$, $X_{\mathcal{K}^\perp} \in \mathcal{K}^\perp$ and similarly for Y. The variable $\kappa \in \check{\mathcal{K}}$ had been omitted for brevity; RX stands for the action of $R(\kappa)$ on $X \in \mathcal{A}$ and so on. It is often more convenient to verify (5.5) case by case for the different choices of X and Y, than to inspect all components of the threefold tensor product in (5.4). It is well known that (5.4) is also equivalent to $\mathrm{CDYB}(r \pm C\hat{I}) = 0$.

Let now $\mathcal{G}$ be a finite-dimensional complex, self-dual Lie algebra with the invariant 'scalar product' denoted as $B(\xi, \eta)$ for any $\xi, \eta \in \mathcal{G}$. Let us suppose that μ is an automorphism of $\mathcal{G}$ of order $N \in \mathbb{N}$, $\mu^N = \mathrm{id}$, that has nonzero fixed points and satisfies $B(\mu(\xi), \mu(\eta)) = B(\xi, \eta)$. (The last two properties of μ are automatic if $\mu = \mathrm{id}$ or $\mathcal{G}$ is simple, which are included as special cases.) Then $\mathcal{G}$ can be decomposed as a direct sum of the eigensubspaces of μ as

$$\mathcal{G} = \oplus_{a \in \mathcal{E}_\mu} \mathcal{G}_a \qquad \mathcal{E}_\mu \subset \{0, 1, \dots, (N-1)\} \tag{5.6}$$

$$\mathcal{G}_a := \left\{ \xi \in \mathcal{G} \mid \mu(\xi) = \exp\left(\frac{\mathrm{i}a2\pi}{N}\right)\xi \right\} \neq \{0\}. \tag{5.7}$$

Note that $\mathcal{G}_a$ is perpendicular to $\mathcal{G}_b$ with respect to the bilinear form B unless $a + b = N$ or $a = b = 0$, which implies that if a nonzero a belongs to the index set $\mathcal{E}_\mu$ then so does $(N - a)$, and $\mathcal{G}_0 \neq \{0\}$ is a self-dual subalgebra of $\mathcal{G}$. The *twisted loop algebra* $\ell(\mathcal{G}, \mu)$ is by definition the subalgebra of $\mathcal{G} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ generated by elements of the form

$$\xi^{n_a} := \xi \otimes \lambda^{n_a} \qquad \text{with} \quad \xi \in \mathcal{G}_a \quad n_a = a + m_a N \quad m_a \in \mathbb{Z}. \tag{5.8}$$

The 'affine Lie algebra' $\mathcal{A}(\mathcal{G}, \mu)$ is given by

$$\mathcal{A}(\mathcal{G}, \mu) := \ell(\mathcal{G}, \mu) \oplus \mathbb{C}d \oplus \mathbb{C}\hat{c} \tag{5.9}$$

with the Lie bracket of its generators defined as

$$[\xi^{n_a}, \eta^{p_b}] = [\xi, \eta]^{n_a + p_b} + n_a \delta_{n_a, -p_b} B(\xi, \eta)\hat{c} \qquad \forall \xi \in \mathcal{G}_a \quad \eta \in \mathcal{G}_b \tag{5.10}$$

$$[d, \xi^{n_a}] = n_a \xi^{n_a} \qquad [\hat{c}, d] = [\hat{c}, \xi^{n_a}] = 0. \tag{5.11}$$

A nondegenerate scalar product $\langle\ ,\ \rangle$ can be defined on $\mathcal{A}(\mathcal{G}, \mu)$ by setting

$$\langle \xi^{n_a}, \eta^{p_b} \rangle = \delta_{n_a, -p_b} B(\xi, \eta) \qquad \langle \hat{c}, d \rangle = 1 \quad \langle d, \xi^{n_a} \rangle = \langle \hat{c}, \xi^{n_a} \rangle = 0. \tag{5.12}$$

Notice that $\mathcal{A}(\mathcal{G}, \mu)$ is graded by the eigenvalues of $\mathrm{ad}\, d$,

$$\mathcal{A}(\mathcal{G}, \mu) = \oplus_{n \in (\mathcal{E}_\mu + N\mathbb{Z})} \mathcal{A}(\mathcal{G}, \mu)_n \tag{5.13}$$

whereby we obtain a decomposition of the type (5.1) with

$$\mathcal{K} := \mathcal{A}(\mathcal{G}, \mu)_0 = \mathcal{G}_0 \oplus \mathbb{C}d \oplus \mathbb{C}\hat{c} \qquad \mathcal{K}^\perp = \oplus_{n \in (\mathcal{E}_\mu + N\mathbb{Z}) \setminus \{0\}} \mathcal{A}(\mathcal{G}, \mu)_n. \tag{5.14}$$

We regard $\mathcal{G}_0$ as a subspace of $\mathcal{A}(\mathcal{G}, \mu)$ by identifying $\xi \in \mathcal{G}_0$ with $\xi \otimes \lambda^0 \in \mathcal{A}(\mathcal{G}, \mu)$; and now we set $\mathcal{A} := \mathcal{A}(\mathcal{G}, \mu)$ for brevity.

To describe the dynamical r-matrix of our interest, $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$, we parametrize the general element $\kappa \in \mathcal{K} = \mathcal{A}_0$ as

$$\kappa = \omega + kd + l\hat{c} \qquad \omega \in \mathcal{G}_0 \quad k, l \in \mathbb{C}. \tag{5.15}$$

Let f and F be the following complex analytic functions:

$$f : z \mapsto \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z} \qquad F : z \mapsto \frac{1}{2}\coth\frac{z}{2}. \tag{5.16}$$

By definition, $R(\kappa)$ ($\kappa \in \check{\mathcal{K}}$) is given by the collection of the finite-dimensional linear operators

$$R(\kappa)|_{\mathcal{A}_0} := f((\mathrm{ad}\,\kappa)_0) \qquad R(\kappa)|_{\mathcal{A}_n} := F((\mathrm{ad}\,\kappa)_n) \quad \forall n \in (\mathcal{E}_\mu + N\mathbb{Z}) \setminus \{0\} \tag{5.17}$$

where $(\mathrm{ad}\,\kappa)_n := \mathrm{ad}\,\kappa|_{\mathcal{A}_n}\ \forall n \in (\mathcal{E}_\mu + N\mathbb{Z})$. These finite-dimensional operators are given analogously to (4.5). Therefore, for them to be well defined, the spectrum of $\mathrm{ad}\,\kappa$ on $\mathcal{A}_n$ must not contain any pole of the respective functions f (for $n = 0$) and F (for $n \neq 0$). This condition could be spelled out explicitly by using that for $\xi \in \mathcal{G}_a$ and $n_a = (a + mN)$ with $m \in \mathbb{Z}$ $(\mathrm{ad}\,\kappa)\xi^{n_a} = ((kn_a + \mathrm{ad}\,\omega)\xi)^{n_a}$. This relation translates the condition on the spectrum of $\mathrm{ad}\,\kappa$ into a condition on the spectrum of $\mathrm{ad}\,\omega$ on the $\mathcal{G}_a$. It is not difficult to see from this that $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ is indeed well defined on a domain of the form

$$\check{\mathcal{K}} = \{\kappa = \omega + kd + l\hat{c} \mid l \in \mathbb{C},\ k \in (\mathbb{C} \setminus \mathbb{R}\mathrm{i}),\ \omega \in \mathcal{B}_k\} \tag{5.18}$$

where $\mathcal{B}_k \subset \mathcal{G}_0$ is an open subset depending on k for which the above conditions hold (for a more explicit description, see [14]). The corresponding map $r : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ is antisymmetric and is $\mathcal{K}$-equivariant. Its interest is due to the following statement.

Proposition 2. *The dynamical r-matrix $R : \check{\mathcal{K}} \to \mathrm{End}(\mathcal{A})$ defined by equations (5.17) with (5.16) on a domain of the form in (5.18) satisfies the operator version (5.5) of the CDYBE with $C = \frac{1}{2}$.*

The verification of the proposition is not difficult, but it is rather long. It is presented in [14]. As an equivalent statement, it follows that the r-matrices $r^\pm : \check{\mathcal{K}} \to \mathcal{A} \otimes \mathcal{A}$ that are associated with the operators $R^\pm := R \pm \frac{1}{2}I$ satisfy the CDYBE (1.1). Formally, these r-matrices can be thought of as special cases of (4.8). Our point is that they are well defined in the infinite-dimensional situation considered here. It may also be checked that these r-matrices are $\mathcal{K}$-equivariant, the condition in terms of R being $(\nabla_{[T,\kappa]}R)(\kappa) = [\mathrm{ad}\,T, R(\kappa)], \forall T \in \mathcal{K}, \kappa \in \check{\mathcal{K}}$.

We finish by a remark on a reinterpretation of the above $\mathcal{A} \otimes \mathcal{A}$-valued r-matrices as spectral-parameter-dependent r-matrices. It is well known that spectral-parameter-dependent $\mathcal{G} \otimes \mathcal{G}$-valued r-matrices may be obtained by applying evaluation homomorphisms to $\ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu)$-valued r-matrices. In the context of dynamical r-matrices, Etingof and Varchenko [5] used this method to recover Felder's elliptic dynamical r-matrices [4] from the standard trigonometric dynamical r-matrices of the affine Lie algebras based on the complex simple Lie algebras. In fact, the same procedure can be applied to the more general family of dynamical r-matrices given by proposition 2. The first step is to set $\hat{c}$ to zero and fix the value of k. Thereby $r^\pm(\kappa) \in \mathcal{A} \otimes \mathcal{A}$ become $\ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu)$-valued dynamical r-matrices, $r^{k,\pm} : \mathcal{B}_k \to \ell(\mathcal{G}, \mu) \otimes \ell(\mathcal{G}, \mu)$, which depend parametrically on k. By using the standard evaluation homomorphisms along the lines of [5], $r^{k,\pm}$ are then converted into $\mathcal{G} \otimes \mathcal{G}$-valued spectral-parameter-dependent dynamical r-matrices, $r^{k,\pm}(\omega, z)$. The final result can be described as follows. Introduce the functions $\chi_a(w, z|\tau)$ of the complex variables w, z by

$$\chi_a(w, z|\tau) := \exp\left(\frac{2\pi \mathrm{i} a z}{N}\right)\left(\frac{1}{2\pi\mathrm{i}}\frac{\theta_1(\frac{w}{2\pi\mathrm{i}} + \frac{a}{N}\tau + z|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(\frac{w}{2\pi\mathrm{i}} + \frac{a}{N}\tau|\tau)} - \frac{\delta_{a,0}}{w}\right) \tag{5.19}$$

where θ_1 is the standard theta-function[4], and let T_α, T^β denote dual bases of $\mathcal{G}$. In fact, one obtains the r-matrix $r^{k,+}(\omega, z) = B(T_\alpha, \mathcal{R}(\omega, z|\tau)T_\beta)T^\alpha \otimes T^\beta$ where $\mathcal{R}(\omega, z|\tau) \in \mathrm{End}(\mathcal{G})$ is defined by

$$\mathcal{R}(\omega, z|\tau)|_{\mathcal{G}_a} := \chi_a(\mathrm{ad}\,\omega, z|\tau) \qquad \text{on} \quad \mathcal{G}_a \quad \forall a \in \mathcal{E}_\mu \quad \omega \in \mathcal{B}_k \subset \mathcal{G}_0. \tag{5.20}$$

The relation between the parameters k and τ reads as $\tau := \frac{kN}{2\pi\mathrm{i}}$, where we assumed that $\Re(k) < 0$. The derivation of this formula is contained in [14]. If $\mathcal{G}$ is a simple Lie algebra and μ is an inner automorphism corresponding to a Coxeter element in the Weyl group, then the spectral-parameter-dependent r-matrices given by (5.20) are equivalent to Felder's elliptic dynamical r-matrices, as expected upon comparison with section 4.6 in [5]. In the general case, the r-matrices provided by (5.17) and (5.20) appear to be new.

[4] We have $\theta_1(z|\tau) = \vartheta_1(\pi z|\tau)$ with ϑ_1 in [17].

6. Discussion

In this paper we have pointed out that solutions of the CDYBE can be mapped to other solutions by Dirac reductions of their underlying Poisson–Lie groupoids, if the conditions given in (3.1) and (3.9) are satisfied. Among the possible applications of proposition 1, we mentioned the antisymmetric solutions that are obtained as reductions of the zero r-matrix[5], and the class of r-matrices given by (4.7) that are reductions of the canonical r-matrix (4.4). Many of these r-matrices are already known, but the class defined by (4.7) contains new examples, too. Although our construction works rigorously only in the finite-dimensional case, some interesting r-matrices that result from it turned out to be well defined in certain infinite-dimensional situations as well. In particular, we exhibited a family of dynamical r-matrices on affine Lie algebras based on arbitrary finite-dimensional self-dual Lie algebras. The r-matrices provided by proposition 2 are in the general case new, and this family includes as special cases those trigonometric r-matrices of [5] that become Felder's elliptic dynamical r-matrices [4] upon evaluation homomorphisms.

The 'canonical' r-matrices (4.4) appear in the description of the chiral sectors of the classical WZNW model in association with any finite-dimensional self-dual Lie algebra [8]. Some of their reductions to self-dual subalgebras have been considered in this context in [19]. We also wish to mention the paper [20], where the effect of a Dirac reduction of the chiral WZNW phase space on constant exchange r-matrices has been studied. The derivation of the trigonometric r-matrix (4.9) contained in this paper served as one of our original motivations, but its status in terms of the geometric interpretation of the dynamical r-matrices is still to be understood. Another open question is whether the general family of r-matrices given by proposition 2 can be used to encode the Poisson brackets of generalized versions of the WZNW model. As candidates, we have in mind both the WZNW models formally obtained by replacing the finite-dimensional WZNW group with an extended affine Lie group [21], which is useful in the theory of soliton equations [22], and the intriguing quasitriangular WZNW model recently introduced by Klimcik [23]. Felder's r-matrices are already known to play this role in these models.

Somewhat implicitly (as a special case of theorem 3.14), the canonical r-matrices (4.4) first appeared in [5]. In their explicit form, they were found independently in the papers [7, 8]. More precisely, in [5, 7] the assumption that the underlying Lie algebra is reductive was used, while [8] provides an indirect approach to these r-matrices on any self-dual Lie algebra. A direct proof of the statement that (4.4) satisfies the CDYBE for any finite -dimensional self-dual Lie algebra is given in [14]. For a different proof in a generalized case, see [13]. The relationship between the generalizations of the r-matrices (4.4) constructed in [13] and the r-matrices given by our proposition 2 is explained in [14].

It would be interesting to develop the quantization of the r-matrices defined by (4.4). If that was found, one could in principle obtain the quantizations of the reduced r-matrices in (4.7) by means of appropriate quantum Hamiltonian reductions. We also wish to clarify whether the canonical r-matrices make sense in cases for which the dynamical variable λ lies in a 'suitable' *infinite-dimensional* self-dual Lie algebra $\mathcal{A}$. Formula (4.5) itself is well defined [15] if $\mathcal{A}$ is a Banach space and ad λ is a bounded operator on it.

We hope to be able to return to the above questions in the future.

[5] Note added: we learned after submitting this paper that these dynamical r-matrices, given by $\mathcal{D}$ in proposition 1, have also been found recently in [18] by using a different method.

Acknowledgments

LF wishes to thank J Balog for useful discussions and for comments on the manuscript. This investigation was supported in part by the Hungarian Scientific Research Fund (OTKA) under T034170, T029802, T030099 and M028418.

References

[1] Etingof P and Schiffmann O 1999 Lectures on the dynamical Yang–Baxter equations *Preprint* math.QA/9908064
[2] Gervais J-L and Neveu A 1984 *Nucl. Phys.* B **238** 125
Cremmer E and Gervais J-L 1990 *Commun. Math. Phys.* **134** 619
[3] Balog J, Dąbrowski L and Fehér L 1990 *Phys. Lett.* B **244** 227
[4] Felder G 1994 *Proc. Int. Congr. Math. (Zürich)* pp 1247–55
Felder G and Wieczerkowski C 1996 *Commun. Math. Phys.* **176** 133
[5] Etingof P and Varchenko A 1998 *Commun. Math. Phys.* **192** 77
[6] Drinfeld V G 1983 *Sov. Math.–Dokl.* **27** 68
[7] Alekseev A and Meinrenken E 2000 *Invent. Math.* **139** 135
[8] Balog J, Fehér L and Palla L 1999 *Phys. Lett.* B **463** 83
Balog J, Fehér L and Palla L 2000 *Nucl. Phys.* B **568** 503
[9] Figueroa-O'Farrill J M and Stanciu S 1996 *J. Math. Phys.* **37** 4121
[10] Sfetsos K 1994 *Int. J. Mod. Phys.* A **9** 4759
[11] Weinstein A 1988 *J. Math. Soc. Japan* **4** 705
[12] Dirac P A M 1950 *Can. J. Math.* **2** 147
[13] Etingof P and Schiffmann O 2001 *Math. Res. Lett.* **8** 157
[14] Fehér L and Pusztai B G 2001 *Preprint* in preparation
[15] Dunford N and Schwartz J T 1958 *Linear Operators, I. General Theory* (New York: Interscience)
[16] Morozov A Y, Perelomov A M, Rosly A A, Shifman M A and Turbiner A V 1990 *Int. J. Mod. Phys.* A **5** 803
[17] Whittaker E T and Wattson G N 1927 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)
[18] Xu P 2001 Quantum dynamical Yang–Baxter equation over a nonabelian base *Preprint* math.QA/0104071
[19] Fehér L 2001 Dynamical r-matrices and the chiral WZNW phase space *Proc. 'Group23' Int. Coll.* at press (Fehér L 2001 *Preprint* math-ph/0104027)
[20] Fateev V A and Lukyanov S L 1992 *Int. J. Mod. Phys.* A **7** 853
[21] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1991 *Phys. Lett.* B **254** 372
[22] Ferreira L A, Miramontes J L and Guillen J S 1995 *Nucl. Phys.* B **449** 631
[23] Klimcik C 2001 Quasitriangular WZW model *Preprint* hep-th/0103118